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# Gust–Shock Interaction in Transonic Small-Disturbance Flow

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**The sound generated by interaction of small-amplitude convected disturbances with an attached shock wave in transonic small-disturbance flow is analyzed using Goldstein's decomposition of unsteady compressible flow (Goldstein, M. E., "Unsteady Vortical and Entropic Disturbances of Potential Flows Round Arbitrary Obstacles," *Journal of Fluid Mechanics*, Vol. 89, 1978, pp. 433–468). The equations obtained by linearizing the Euler equations about the nonuniform mean flow provide a framework that enables the calculation of the vortical, entropic, and acoustic waves generated by the gust–shock interaction. The lateral stretching of disturbances that is characteristic of transonic small-disturbance flow implies that the shock wave is long relative to the $\mathcal{O}(1)$ or smaller gust wavelength and that its strength and obliqueness angle vary slowly along its length. This leads to inner regions near the shock where the jump relations are solved and to an outer region where the shock's finite length and variation in strength determine the far-field acoustic radiation. The theory is applied to obtain numerical results for the radiated acoustic power and directivity patterns for a shock wave attached to an infinite curved surface. It is found that sound is radiated primarily at the orientation angle of the incident gust, especially for high reduced frequencies.**

## I. Introduction

COMPREHENSIVE noise prediction schemes for turbomachinery applications must incorporate the effects of interaction between convected vortical and entropic disturbances and blade rows; for many modern aeroengines this acoustic study is especially relevant when transonic mean-flow conditions are assumed. An important aspect of gust–blade interaction at supercritical Mach numbers is the influence of shock waves, not only on the propagation of existing sound waves but also as sources of sound when they interact with the convected disturbances. The latter interaction between small-amplitude gusts and a shock wave in transonic small-disturbance (TSD) flow is the subject of this paper. A critical step in modeling this interaction is the suitable decomposition of the unsteady velocity field into potential and vortical components as specified by Goldstein.[1]

Previous results in this area by Ribner,[2] Moore,[3] and Hardy and Atassi[4] consider small-amplitude solutions to the unsteady Euler equations linearized about a base flow that is uniform on either side of a normal shock of infinite length and of constant strength. The usual shock jump relations relate the plane vorticity, sound, and attenuated pressure waves downstream of the shock to those upstream. Ribner[5] and Mahesh et al.[6] extend these results to study the interaction of isotropic fields of acoustic and vortical waves with a shock wave. Williams[7] also linearizes about uniform flow; the analysis applies for weak shocks in a purely potential flow (i.e., no vortical or entropic disturbances), though it allows for more complicated unsteady response by imposing a boundary condition that represents deformation of an airfoil. Studies that incorporate mean-flow distortion or, indeed, that consider solutions to the nonlinear, time-dependent Euler equations rely on numerical methods. For example, Verdon and Caspar[8] apply a code based on the linearization of the Euler equations about the steady flow through a cascade. Rusak et al.[9] modify the unsteady TSD equation to include small-amplitude vortical components and solve it using a mixed-type difference scheme.

The analyses by Ribner[2] and Hardy and Atassi[4] on gust–shock interaction give valuable insights into the mechanisms that control the generation of convected and propagated waves. In this paper, their results are extended to take into account two features that are significant in turbomachinery applications. First, the mean flow on either side of the shock wave is generally nonuniform, directly influencing the interactions at the shock and leading to distortion of the incident and outgoing disturbances. The analysis presented here applies for mean flow that solves the two-dimensional TSD equation (a full three-dimensional analysis is performed on the unsteady perturbation to this mean). Second, the shock wave is finite in length, its strength varies along its length, and it may be attached to a surface at one or both ends. Also, consistent with our modeling of the TSD steady flow, we permit the shock wave to be slightly curved and oblique. These features have important consequences for the radiation in the far field; gust–shock interaction for an infinite shock of constant strength induces unattenuated acoustic plane waves, but the finite extent of the interaction allows for cylindrical decay.

Although incident acoustic disturbances may easily be included in the theory, we focus on harmonic vortical and entropic disturbances that are convected with and distorted by the mean flow before interacting with the weak shock.

For uniform mean flow, the appropriate splitting of the unsteady disturbances into potential and solenoidal parts[10] provides a means of solving the equations of motion on either side of the shock wave. Coupled with the boundary condition at the shock imposed by the jump relations, this makes solution of the downstream flow in terms of the upstream quantities possible. For nonuniform mean flow, the classical splitting theorem does not apply. Instead, the decomposition given by Goldstein[1] for irrotational mean flow is used, although the splitting does not result in components that are purely irrotational and divergence free.

The problem is presented as follows. First, in Sec. II we formulate the shock-wave problem by outlining how Goldstein's decomposition[1] and the shock jump relations are to be applied in potential-stream function coordinates. The problem is then solved in Sec. III. Asymptotics corresponding to the TSD assumption are imposed, solutions are found for the problems at the relevant asymptotic orders, and, finally, the far-field radiation is calculated. Numerical results are given for a shock wave over a curved surface in Sec. IV.

## II. Problem Formulation

### A. Equations of Motion

In our analysis we assume the unsteady flow to be an $\mathcal{O}(\epsilon)$ perturbation to a two-dimensional, steady TSD flow for some $\epsilon \ll 1$. In turn, the mean flow is a small disturbance to a uniform flow of speed $U_\infty$ parallel with the $x_*$ axis of our $(x_*, y_*)$ coordinate system. The freestream Mach number is $M_\infty = U_\infty / a_\infty$, where $a_\infty$ is the speed of sound and $p_\infty$, $\rho_\infty$, and $S_\infty$ denote the far-field pressure, density, and entropy, respectively (upstream as well as downstream of the

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shock). The quantity $\beta_\infty^2 = 1 - M_\infty^2$ is small but may be positive or negative. We assume that the mean flow is that of an isentropic perfect gas with $S_\infty = 0$, over an infinite, curved surface with an $\mathcal{O}(\delta)$ displacement from the $x_*$ axis for some $\delta \ll 1$. The corresponding steady flow is described by the velocity potential $\phi$ and stream function $\psi$, both normalized by $U_\infty b$, where $b$ is some characteristic length of the surface shape in the $x_*$ direction. We define the normalized coordinates $x = x_*/b$ and $y = y_*/b$ and write $\phi = x + \phi'$. Then, for TSD flow, the constant $\beta_\infty^2$ and the normalized disturbance potential $\phi'$ are taken to have the preferred magnitude $\mathcal{O}(\delta^{2/3})$. The latter is rewritten as $\phi' = \delta^{2/3}\varphi'$, where $\varphi'$ satisfies the TSD equation

$$\left(K - (\gamma + 1)\frac{\partial \varphi'}{\partial x}\right)\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial \tilde{y}^2} = 0 \tag{1}$$

and where $\tilde{y} = \delta^{1/3} y$, $\gamma = c_p/c_v$ is the ratio of specific heats, and $K = \beta_\infty^2 M_\infty^{-1} \delta^{-2/3}$ is the transonic similarity variable.[11] The normalized perturbation speed,

$$q = \frac{\partial \phi'}{\partial x} = \mathcal{O}\left(\delta^{\frac{2}{3}}\right) \tag{2}$$

will also be used to describe the mean flow. We need not specify the boundary condition along the surface, but merely note that it is consistent with uniform flow in the far field and a shock wave along $x = \eta_0(\tilde{y})$ for some function $\eta_0$ defined on $0 \le \tilde{y} \le \tilde{y}_s$ (between the shock extremes). Flow quantities upstream and downstream of the shock will be distinguished by the subscripts 1 and 2, respectively, and the shock strength is given by the jump $\Delta q = q_2 - q_1 = \mathcal{O}(\delta^{2/3})$ of the perturbation speed. Note that an important feature of TSD flow is that the shock characteristics vary with $\tilde{y}$, that is, they vary on the scale $b/\delta^{1/3}$, so that the angle of the shock at any point with respect to the $y$ axis is $\mathcal{O}(\delta^{1/3})$.

The total local flow speed, velocity, Mach number, speed of sound, pressure, density, and entropy are denoted by $U$, $\boldsymbol{u}$, $M$, $a$, $p$, $\rho$, and $S$, respectively. The subscript 0 denotes mean-flow properties, and a prime denotes the unsteady perturbation; for example,

$$p(x, y, z, t) = p_0(x, y) + \epsilon p'(x, y, z, t) + \mathcal{O}(\epsilon^2)$$

where $t$ is time normalized by $b/U_\infty$. So that the unsteady flow may be linearized about the mean flow, we require $0 < \epsilon \ll \delta^{2/3} \ll 1$.

In the following analysis we consider Goldstein's[1] formulation for inviscid linear disturbances of a two-dimensional irrotational mean flow. In this formulation, small-amplitude vortical and entropic disturbances are assumed to be present in the uniform flow far upstream. The sound generated by interaction between these convected disturbances and obstacles downstream (in this case the shock wave) is described by an inhomogeneous convected-wave equation with variable coefficients. We follow Goldstein's decomposition and write the unsteady velocity in the form

$$\boldsymbol{u}' = \nabla G' + \boldsymbol{v}' \tag{3}$$

where

$$\boldsymbol{v}' = (s'/2c_p)\boldsymbol{u}_0 + \tilde{\boldsymbol{v}}' \tag{4}$$

and $\tilde{\boldsymbol{v}}'$ and $G'$ satisfy the equations

$$\frac{D_0 \tilde{\boldsymbol{v}}'}{Dt} + \tilde{\boldsymbol{v}}' \cdot \nabla \boldsymbol{u}_0 = 0 \tag{5}$$

$$\frac{D_0}{Dt}\left(\frac{1}{a_0^2}\frac{D_0 G'}{Dt}\right) - \frac{1}{\rho_0}\nabla \cdot (\rho_0 \nabla G') = \frac{1}{\rho_0}\nabla \cdot (\rho_0 \boldsymbol{v}') \tag{6}$$

respectively. The operator $D_0/Dt = \partial/\partial t + \boldsymbol{u}_0 \cdot \nabla$ is the substantial derivative with respect to the mean flow. The potential $G'$ describes the nonvortical components of the unsteady field, including the acoustic waves, and completely determines the unsteady pressure through the relation

$$p' = -\rho_0 \frac{D_0 G'}{Dt} \tag{7}$$

The velocity $\boldsymbol{v}'$ contains the vorticity fluctuations (the upstream disturbances as well as vorticity generated farther downstream), though in general is not divergence free, so that Eq. (3) does not correspond to the classical decomposition into potential and solenoidal parts. Finally, the unsteady entropy and pressure are determined by

$$\frac{D_0 s'}{Dt} = 0, \qquad \rho' = \frac{p'}{a_0^2} - \frac{\rho_0 s'}{c_p} \tag{8}$$

and the rigid-surface boundary condition is

$$\boldsymbol{n} \cdot (\nabla G' + \boldsymbol{v}') = 0 \tag{9}$$

for the vector $\boldsymbol{n}$ normal to the surface.

Following Kerschen and Balsa,[12] general solutions to Eqs. (5) and (8) are obtained by assuming a harmonic time dependence with frequency $\omega$ and transforming from $(x_*, y_*, z_*)$ to the potential and stream function coordinates $(\phi, \psi, z)$ with corresponding metrics $m_\phi = bU_\infty/U_0$, $m_\psi = b\rho_\infty U_\infty/\rho_0 U_0$, and $m_z = b$. The result is

$$\tilde{v}_t' = \frac{U_\infty}{U_0} h_a(\psi, z)\exp[ik(\phi + g)]$$

$$\tilde{v}_n' = \frac{\rho_0 U_0}{\rho_\infty U_\infty}\left(\frac{\partial g}{\partial \psi} h_a(\psi, z) + h_b(\psi, z)\right)\exp[ik(\phi + g)]$$

$$\tilde{v}_3' = h_c(\psi, z)\exp[ik(\phi + g)]$$

$$s' = 2c_p h_d(\psi, z)\exp[ik(\phi + g - t)] \tag{10}$$

where $k = \omega b/U_\infty$ is the aerodynamic reduced frequency and $U_\infty(\tilde{v}_t', \tilde{v}_n', \tilde{v}_3')e^{-ikt}$ represents the velocity $\tilde{\boldsymbol{v}}'$ decomposed into the directions parallel with the $\phi$, $\psi$, and $z$ axes. The functions $h_{a-d}$ depend on $\psi$ and $z$ and are determined by far-field boundary conditions upstream and by the jump relations at the shock wave. The function $g$ is the drift function, a measure of the cumulative distortion of the gust by the nonuniform mean flow, and for a potential flow that is a small perturbation to uniform flow, it is approximated by

$$g(\phi, \tilde{\psi}) \sim 2[\phi'(-\infty, 0) - \phi'(\phi, \tilde{\psi})] \tag{11}$$

selected to vanish at upstream infinity. Because $\phi = x + \mathcal{O}(\delta^{2/3})$ and $\psi = y + \mathcal{O}(\delta^{2/3})$, we may write $g$ and the other mean-flow quantities, $q$ and $\phi'$, as functions of $\phi$ and $\tilde{\psi} = \delta^{1/3}\psi$, consistent with $g(x, \tilde{y}) = g(\phi, \tilde{\psi}) + \mathcal{O}(\delta^{4/3})$. The unsteady potential $G'$ is the remaining unknown, and it is found by solving Eq. (6) with the appropriate boundary condition at the surface, a far-field radiation condition, and with the jump relations at the shock.

### B. Shock Jump Relations

The functions $\phi$ and $\psi$ are continuous across the shock wave and the mean position of the shock in $(\phi, \psi)$ space is $\phi = \eta_0(\tilde{\psi})$ between $0 \le \tilde{\psi} \le \tilde{y}_s$ to sufficient $\mathcal{O}(\delta^{2/3})$ accuracy. Interaction with the incident disturbance will cause the shock to undergo $\mathcal{O}(\epsilon)$ oscillations about its mean position, though the linearization in $\epsilon$ allows us to apply the jump relations at $\phi = \eta_0$. The equations of motion and jump relations are solved in $(\phi, \psi)$ space, but it is convenient to express the displacement of the shock as a function of the normalized physical coordinates; the shock's instantaneous position is therefore given by

$$x = \eta(y, z, t) = \eta_0(\tilde{y}) + \epsilon\eta'(y, z)e^{-ikt} \tag{12}$$

for some unknown function $\eta'$. We make use of the orthogonal vectors

$$\boldsymbol{\xi} = \left(1, -\frac{\partial \eta}{\partial y}, -\frac{\partial \eta}{\partial z}\right), \qquad \boldsymbol{\tau} = \left(\frac{\partial \eta}{\partial y}, 1, 0\right)$$

$$\boldsymbol{\zeta} = \left(\frac{\partial \eta}{\partial z}, -\frac{\partial \eta}{\partial z}\frac{\partial \eta}{\partial y}, 1 + \left(\frac{\partial \eta}{\partial y}\right)^2\right)$$

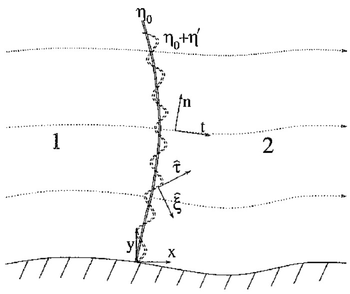


**Fig. 1 Shock wave and coordinate systems.**

the first of which is normal to the shock wave. The variables $\hat{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\tau}}$, and $\hat{\boldsymbol{\zeta}}$ denote the corresponding unit vectors. Next, let $\chi_0 \sim v_0/U_0$ be the angle between the $x$ axis and the mean flow at any position along the shock. Here, $v_0$ is the vertical speed of the mean flow. Then, if $\boldsymbol{u}' = (u'_t, u'_n, u'_3)$ is the unsteady velocity in components tangential and normal to the local mean flow, the unsteady velocity relative to the shock in physical coordinates is

$$\boldsymbol{u}'_{\text{rel}} = -\begin{pmatrix}1\\0\\0\end{pmatrix}\frac{\partial \eta}{\partial t} + \begin{pmatrix}\cos\chi_0 & \sin\chi_0 & 0\\ -\sin\chi_0 & \cos\chi_0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}u'_t\\u'_n\\u'_3\end{pmatrix} \tag{13}$$

In coordinates normal and tangential to the shock (Fig. 1), this relative velocity becomes

$$(u'_\xi, u'_\tau, u'_\zeta) \equiv (\boldsymbol{u}'_{\text{rel}}\cdot\hat{\boldsymbol{\xi}}, \boldsymbol{u}'_{\text{rel}}\cdot\hat{\boldsymbol{\tau}}, \boldsymbol{u}'_{\text{rel}}\cdot\hat{\boldsymbol{\zeta}}) \tag{14}$$

The perturbed shock may be regarded as behaving in a locally quasi-steady manner, so that the usual Rankine–Hugoniot relations apply. We use them in the form[13]

$$\frac{[u'_\xi]}{a_1} = -\frac{2\mu}{(2\gamma)^{\frac{1}{2}}[(\gamma+1)\mu + 2\gamma]^{\frac{1}{2}}} \tag{15}$$

$$[u'_\tau] = 0 \tag{16}$$

$$[u'_\zeta] = 0 \tag{17}$$

$$\frac{u'_{\xi 1}}{a_1} = \left(1 + \frac{(\gamma+1)\mu}{2\gamma}\right)^{\frac{1}{2}} \tag{18}$$

$$\frac{[s']}{c_v} = \ln(1+\mu) - \gamma\ln\left(1 + \frac{2\mu}{(\gamma-1)\mu + 2\gamma}\right) \tag{19}$$

where $\mu = [p]/p_1$ and the jump in any quantity $X$ across the shock is indicated by $[X] = X_2 - X_1$.

### C. Incident Gust

We complete the formulation of the shock-wave problem as follows. The functions $h_{a-d}$ are given the additional subscript 1 or 2 to correspond to solutions upstream and downstream of the shock, respectively. [The subscript 1 has no meaning for values $\tilde{\psi} > \tilde{y}_b$ (i.e., for streamlines that do not pass through the shock), in which case we write $h_{a1} = h_{a2} = h_a$ and similar relations for the other three functions that define the velocity $\tilde{\boldsymbol{v}}'$ and entropy $s'$.] An incident harmonic gust is then defined by fixing

$$h_{a1} = A_t^* e^{ik\sigma_g}, \qquad h_{b1} = A_n e^{ik\sigma_g}$$

$$h_{c1} = A_3 e^{ik\sigma_g}, \qquad h_{d1} = B e^{ik\sigma_g} \tag{20}$$

in Eq. (10), so that far upstream it consists of the plane waves

$$\boldsymbol{v}' \sim U_\infty(A_t, A_n, A_3)\exp[ik(\phi + \sigma_g) - ikt]$$

$$s' \sim 2c_p B\exp[ik(\phi + \sigma_g) - ikt] \tag{21}$$

as $\phi \to -\infty$, for arbitrary constants $A_t$, $A_n$, $A_3$, $B$, $k_n$, and $k_3$ that satisfy the relation

$$A_t + A_n k_n + A_3 k_3 = 0 \tag{22}$$

for divergence-free flow. Here, $\sigma_g = k_n\psi + k_3 z$ and $A_t^* = A_t - B$. Relation (22) follows from the (linearized) continuity equation and our assumption that the incident disturbances are convected by a uniform oncoming flow. Effectively, this gives a relation between the functions $h_{a1-d1}$ that define the vortical velocity upstream of the shock. It is not immediately clear that this relation can be generalized to include the functions $h_{a2-d2}$ for the flow downstream of the shock, where we expect propagated waves in addition to the convected disturbances. As demonstrated in the Appendix, however, any such relation between the components of the vortical velocity is redundant and merely serves to make decomposition (3) unique. This is not so much an issue when there are no shock waves and the assumption $\lim_{\phi\to-\infty} G' = 0$ renders the problem unique, but rather when the flow contains discontinuities an alternative relation is required to completely specify the decomposition. The relation that is imposed for this purpose is

$$h_a = \frac{i}{k}\left(\frac{\partial h_b}{\partial \psi} + \frac{\partial h_c}{\partial z}\right) - h_d \tag{23}$$

which reduces to the condition of divergence-free convected flow in the limit $\delta \to 0$.

What remains now is a boundary value problem for $G'$ given by the differential equation (6), the radiation condition, the boundary conditions on the surface, and the boundary conditions along the shock given by the linearization of Eqs. (15–19). The solution depends on the incident gust through the volume-source term on the right-hand side of Eq. (6) and through the boundary conditions. The boundary condition along the shock involves the additional unknown functions $h_{a2}$, $h_{b2}$, $h_{c2}$, $h_{d2}$, and the unsteady shock displacement $\eta'$, which are solved for simultaneously in terms of the incident-gust and mean-flow quantities, using also the final relation (23).

## III. Solution of the Shock-Wave Problem

### A. Simplification for TSD Flow

Because the shock wave is finite in length and the solution upstream of the shock generally depends on the solution downstream, there is no prospect of a closed-form analytical solution to the boundary value problem. Further progress can be made, however, by expanding the equations and their solutions in the perturbation parameter $\delta$.

First, the linearized perfect-gas relations[14] give the mean-flow quantities in terms of the steady disturbance speed $q$. They reduce to

$$U_0 = U_\infty(1+q), \qquad a_0 = a_\infty[1 - (\gamma-1)q/2]$$

$$\rho_0 = \rho_\infty(1-q), \qquad \beta_0^2 = \beta_\infty^2 - (\gamma+1)q \tag{24}$$

for TSD flow to $\mathcal{O}(\delta^{2/3})$ accuracy. The dependence on $t$ and $z$ can be factored out of the equations, and, accordingly, we define the modified unsteady potential $G$ by setting

$$G'(\phi, \psi, z, t) = U_\infty bG(\phi, \psi)\exp(ikk_3 z - ikt) \tag{25}$$

We expand this potential $G$, the vortical flow $\boldsymbol{v}'$, and the unsteady shock displacement $\eta'$ in powers of $\delta^{1/3}$ and use the superscript $(i)$ to denote terms of $\mathcal{O}(\delta^{i/3})$. For example,

$$\eta' = \eta'^{(0)} + \delta^{\frac{1}{3}}\eta'^{(1)} + \delta^{\frac{2}{3}}\eta'^{(2)} + \cdots \tag{26}$$

$$G = G^{(0)} + \delta^{\frac{1}{3}}G^{(1)} + \delta^{\frac{2}{3}}G^{(2)} + \cdots \tag{27}$$

We also assume here and verify later that the transmitted gust is such that we can write

$$h^{(i)}_{a-d}(\psi, z) = \bar{h}^{(i)}_{a-d}(\tilde{\psi})e^{ik\sigma_g}, \qquad i = 0, 1, 2, \ldots \tag{28}$$

or, in other words, that the gust amplitudes are independent of $z$ and vary only slowly in the $\psi$ direction [this is already the case for the incident gust, where $\bar{h}_{a1}^{(0)} = A_t^*$, $\bar{h}_{b1}^{(0)} = A_n$, $\bar{h}_{c1}^{(0)} = A_3$, and $\bar{h}_{d1}^{(0)} = B$, as Eq. (20) shows]. By substituting these solutions into Goldstein's equation[1] and the boundary conditions, a boundary value problem is defined at each order $i$ that should uniquely determine the corresponding $\mathcal{O}(\delta^{i/3})$ solutions.

### B. Leading-Order Solutions

Because the weak shock has only a higher-order influence on the unsteady flow, the values of $\bar{h}_{a2-d2}^{(0)}$ downstream of the shock are the same as the values $\bar{h}_{a1-d1}^{(0)}$ upstream. (A uniqueness issue is addressed in the Appendix.) Consequently, Goldstein's[1] equation (6) and the boundary condition (9) are

$$\frac{\partial^2 G^{(0)}}{\partial \psi^2} + 2ik\frac{\partial G^{(0)}}{\partial \phi} + k^2\left(1 - k_3^2\right)G^{(0)} = 0 \tag{29}$$

$$\left.\frac{\partial G^{(0)}}{\partial \psi}\right|_{\psi=0} = -A_n e^{ik\phi} \tag{30}$$

at leading order, where the right-hand side of Eq. (30) is determined by expanding Eq. (9) for small $\delta$ using Eqs. (4) and (10). It is easily confirmed that

$$G^{(0)} = \left[A_n/k\left(1 + k_3^2\right)^{\frac{1}{2}}\right]\exp\left[ik\phi - k\left(1 + k_3^2\right)^{\frac{1}{2}}\psi\right] \tag{31}$$

is the required potential flow that cancels the gust velocity normal to the surface. We note that this near-wall solution makes no significant contribution to the far-field noise signature owing to its exponential decay.

Because the shock-wave strength $\Delta q$ is $\mathcal{O}(\delta^{2/3})$ small, we expect any sound that is generated to have the same order of magnitude. However, calculations also involve $\mathcal{O}(\delta^{1/3})$ terms because, as the steady shock relations confirm, this is the order of magnitude of the angle of the shock with respect to the vertical. Specifically, the coefficients of $\epsilon^0\delta$ and $\epsilon^0\delta^{2/3}$ in Eqs. (16) and (18) show that

$$\delta^{\frac{1}{3}}\frac{\partial \eta_0^{(1)}}{\partial y} = -\frac{[v_0]}{U_\infty \Delta q} = \pm\left(\bar{M}_0^2 - 1\right)^{\frac{1}{2}} \tag{32}$$

where $\bar{M}_0$ is the average of the local Mach number on both sides of the shock. The coefficient of $\epsilon^0\delta$ in Eq. (18) gives $\partial \eta_0^{(2)}/\partial y = 0$. Although the small angle of the steady shock plays a role in the unsteady jump relations, the relations show that it is not relevant at $\mathcal{O}(\epsilon\delta^{1/3})$ and that the zero solution $G^{(1)} = \bar{h}_{a-d}^{(1)} = 0$ is valid.

For completeness, we note that the shock relations for the freestream flow at $\mathcal{O}(\epsilon^0\delta^0)$ are trivial and that the remaining steady relations at $\mathcal{O}(\epsilon^0\delta^i)$ for $i = 1, 2, 3$ are already satisfied following application of the perfect-gas relations (24).

### C. Local Plane-Wave Representation

The significant interactions that determine the sound generation are found at the next higher order, $\mathcal{O}(\epsilon\delta^{2/3})$. We write the solution as the sum $G^{(2)} = G_c^{(2)} + G_p^{(2)} + G_s^{(2)}$ to distinguish the particular solution $G_p^{(2)}$ induced by volume-source terms, the near-wall solution $G_c^{(2)}$ that cancels the velocity normal to the surface, and a term $G_s^{(2)}$ that completes the solution. $G_s^{(2)}$ includes waves generated at the shock and their reflection and takes into account the (slow) variation of the mean flow with $\tilde{y}$. The asymptotic expansion of Goldstein's[1] equation (6), simplified using Eq. (23) and the zeroth-order solution, specifies

$$\mathcal{L}\left(G_p^{(2)}\right) = 2A_t^*\left(ik\bar{q} + \frac{\partial \bar{q}}{\partial \phi}\right)\exp[ik(\phi + k_n\psi)] \tag{33}$$

$$\mathcal{L}\left(G_c^{(2)}\right) = 2k\left(k\bar{q} + i\frac{\partial \bar{q}}{\partial \phi}\right)G^{(0)} \tag{34}$$

$$\mathcal{L}\left(G_s^{(2)}\right) = 0 \tag{35}$$

where $q$ is rescaled to $\bar{q} = q\,\delta^{-2/3}$ and the operator $\mathcal{L}$ is the transonic convected-wave operator acting on $G^{(0)}$ on the left-hand side of Eq. (29). The source term in Eq. (33) is the leading-order expansion of the right-hand side of Goldstein's[1] equation (6). In Eq. (34) the source term is due to interaction between the nonuniform mean flow and the leading-order solution $G^{(0)}$. We require that $G_c^{(2)}$ satisfies Eq. (34) and that it also cancels the velocity of the (disturbed) gust and of $G_p^{(2)}$ normal to the surface.

The surface curvature is taken to be $\mathcal{O}(\delta)$, and so to the order considered here, the surface may be considered flat (the only influence of the curvature on this problem is through the mean flow). Once more, the near-wall solution $G_c^{(2)}$ has an exponential decay with $\psi$. In the matching between the flows upstream and downstream of the shock, terms with the factor $\exp[-k(1 + k_3^2)^{1/2}\psi]$ may be considered independent of those with the factor $e^{ikk_n\psi}$, and so we need not consider $G_c^{(2)}$ any further.

By contrast, $G_p^{(2)}$ must be included in the analysis of the jump relations at the shock. We write it in the general form $G_p^{(2)} = f(\phi, \tilde{\psi})e^{ikk_n\psi}$, factor out the dependence on $\psi$, and solve the resulting ordinary differential equation. An integration by parts, with endpoint contributions from both sides of the shock and from $\lambda = \phi$, then gives

$$\begin{aligned} G_p^{(2)} = &-\frac{2iA_t^*}{k\left(1 + k_n^2 + k_3^2\right)}\bar{q}(\phi, \tilde{\psi})\exp[ik(\phi + k_n\psi)] \\ &+ \Gamma_{p2}^{(2)}\exp\left\{ik\left[\left(1 - k_n^2 - k_3^2\right)\phi/2 + k_n\psi\right]\right\}H(\phi - \eta_0) \\ &+ \frac{Q\exp(ikk_n\psi)}{k}\int_{-\infty}^{\phi}\frac{\partial \bar{q}(\lambda, \tilde{\psi})}{\partial \lambda}\exp(ik\lambda) \\ &\times\exp\left[ik\left(1 - k_n^2 - k_3^2\right)(\phi - \lambda)/2\right]\mathrm{d}\lambda \end{aligned} \tag{36}$$

where

$$\Gamma_{p2}^{(2)} = \frac{2iA_t^*\Delta\bar{q}(\tilde{\psi})}{k\left(1 + k_n^2 + k_3^2\right)}\exp\left[ik\left(1 + k_n^2 + k_3^2\right)\eta_0(\tilde{\psi})/2\right] \tag{37}$$

$$Q = \frac{1 - k_n^2 - k_3^2}{1 + k_n^2 + k_3^2}iA_t^* = -iA_t^*\cos 2\Theta_g \tag{38}$$

and $H(\cdot)$ is the unit step function. In Eq. (38), the constant $Q$ is expressed in terms of the angle $\Theta_g$ between the gust orientation and the shock, given by $\tan\Theta_g = (k_n^2 + k_3^2)^{-1/2}$.

The particular solution (36) represents unsteady motion due to volume sources in the flow. The integration by parts allows the solution to be expressed as acoustic and hydrodynamic components; the first term represents waves that are convected by the mean flow, whereas the second term satisfies the homogeneous wave equation and represents plane sound waves caused by the discontinuity in the volume sources. The integral that remains also contains acoustic and hydrodynamic components, which may be extracted by further integration by parts. If we make the additional assumption that the the mean flow varies (now in all directions) on a length scale that is large compared to the gust wavelength, that is, $\partial\bar{q}/\partial\phi \ll k$, the pressure disturbances associated with this integral and the first term are asymptotically smaller than the sound pressure waves associated with the second term. With this in mind, we include only the latter in further calculations. Finally, we note that $G_p^{(2)}$ is continuous across the shock (even though its derivative with respect to $\phi$ is not), which is significant when the jump relations are applied.

An additional homogeneous solution $G_s^{(2)}$ is taken to be of the form

$$G_s^{(2)} = \Gamma_{s2}^{(2)}\exp\left\{ik\left[\left(1 - k_n^2 - k_3^2\right)\phi/2 + k_n\psi\right]\right\} \tag{39}$$

again a plane wave propagating downstream of the shock. Upstream, the $\mathcal{O}(\delta^{2/3})$ gust components are prescribed by Eq. (20) so that $\bar{h}_{a1-d1}^{(2)} = 0$ [note from the expansion of Eq. (10) that the gust still has a nonzero component at this order due to distortion by the mean flow], and downstream we calculate $\bar{h}_{a2-d2}^{(2)}$ (along with $\Gamma_2^{(2)}$ and the shock displacement $\eta'^{(2)}$) from the jump relations and Eq. (23).

At any point along the shock we have effectively a one-dimensional problem; the factor $e^{ikk_n\psi}$ can be factored out, and the only remaining variation in the $\psi$ direction is $\mathcal{O}(\delta^{1/3})$ small. The jump relations do involve derivatives in $\phi$, however, and so any dependence of the solutions on $\phi$ must be included. Finally, we mention that in the formulation of this problem, the flow upstream of the shock is completely specified because the gust is given and its distortion and the components of the unsteady potential immediately follow. This implies that the downstream flow exerts no influence on the upstream flow; indeed, at this order in $\delta$ the linearized transonic wave equation, $\mathcal{L}(G)=0$, cannot describe receding waves, and all disturbances travel downstream.

The result of considerable algebra in which the shock relations are expanded to $\mathcal{O}(\epsilon\delta^{2/3})$ is that

$$\Gamma_{s2}^{(2)} = -\left[iA_t^*\Delta\bar{q}(\tilde{\psi})/k\right]\exp\left[ik\left(1+k_n^2+k_3^2\right)\eta_0(\tilde{\psi})/2\right] \quad (40)$$

and $\bar{h}_{a2-d2}^{(2)}=0$. (The solution for the shock displacement $\eta'^{(2)}$ is not included here.) Thus, for every value of the slow coordinate $\tilde{\psi}$, the shock generates a sound wave that locally resembles a plane wave with an amplitude, $\Gamma_{p2}^{(2)}+\Gamma_{s2}^{(2)}$, that is proportional to $\Delta q(\tilde{\psi})$. This amplitude depends on the incident entropy gust and the streamwise component of the incident vortical gust through the factor $A_t^*=A_t-B$. The group velocity corresponding to the phase of these plane waves [see Eq. (39)] shows that the acoustic disturbances propagate at the gust orientation angle $\theta_g=\tan^{-1}k_n$ in the $(\phi,\psi)$ plane. The additional shift involving the mean shock position $\eta_0$ arises naturally from the calculations and ensures that the phases of the incident gust and the sound waves equate at the shock.

The solutions $\bar{h}_{a2-d2}^{(2)}=0$ show that to $\mathcal{O}(\delta^{2/3})$, the velocity $\tilde{\boldsymbol{v}}$ is given by Eq. (10) with $h_{a-d}=h_{a1-d1}$ [see Eq. (20)] on both sides of the shock. Any jumps in the convected components of the unsteady flow are, therefore, given implicitly by the sudden change in the steady-flow quantities in Eq. (10). As expected when shock waves are weak, no additional entropy disturbances are generated. The velocity components $\tilde{v}_n'$ and $\tilde{v}_3'$ are also continuous to the order considered, but the discontinuity of $U_0$ in the first line of Eq. (10) indicates that $\tilde{v}_t'$ jumps across the shock. The sudden refraction of both the entropic and vortical gusts across the shock is represented by the drift function $g$ in the phase.

The sound waves do not satisfy the rigid-wall boundary condition on the $\phi$ axis as they must, but this can be resolved by a simple reflection in $\psi=0$. This step is performed in the next section where the far-field radiation is considered.

### D. Far-Field Radiation

Clearly the sound waves generated at the shock cannot remain plane waves as they propagate away from the shock. The $\mathcal{O}(\delta^{1/3})$ effects introduced by the variation with $\tilde{\psi}$ accumulate, so that the local description of the sound is no longer valid in the far field. We avoid this difficulty by regarding the local plane-wave solutions as a boundary condition for the unsteady potential flow that solves the homogeneous form of Goldstein's[1] equation downstream of the shock. In other words, given the strength of the acoustic disturbances generated at the shock [see Eqs. (37) and (40)], we solve the governing equation for $G$ in the entire region downstream of $\phi=\eta_0$.

At leading order, this problem is given by $\mathcal{L}(G)=0$ with the boundary conditions

$$G[\eta_0(\tilde{\psi}),\tilde{\psi}] = [Q\Delta q(\tilde{\psi})/k]\exp[ik(\eta_0(\tilde{\psi})+k_n\psi)] \quad (41)$$

$$\frac{\partial G}{\partial\psi}(\phi,0)=0, \qquad \phi>\eta_0(0) \quad (42)$$

[The superscript (2) and the subscript 2 have been discarded, though we note that $G$ now represents an $\mathcal{O}(\delta^{2/3})$ function that is valid downstream of the shock only.] By $\mathcal{O}(\delta^{1/3})$ approximation, we can make the change of variable from $\phi$ to $\phi-\eta_0$, and the problem is then solved for $\phi-\eta_0>0$ using conventional methods (see Cannon,[15] for instance). We find $G=\mathcal{G}(\phi,\psi)+\mathcal{G}(\phi,-\psi)$ with

$$\mathcal{G}(\phi,\psi) = \frac{Qe^{-\pi i/4}}{(2\pi k)^{\frac{1}{2}}}\int_0^{\delta^{-\frac{1}{3}}\tilde{y}_s}\frac{\Delta q(\tilde{\tau})}{(\phi-\eta_0)^{\frac{1}{2}}}e^{ik\sigma}\mathrm{d}\tau \quad (43)$$

where

$$\sigma = \frac{\left(1-k_3^2\right)}{2}(\phi-\eta_0)+\frac{(\psi-\tau)^2}{2(\phi-\eta_0)}+\eta_0+k_n\tau \quad (44)$$

and $\eta_0=\eta_0(\tilde{\tau})$ is a function of $\tilde{\tau}=\delta^{1/3}\tau$. The term $\mathcal{G}(\phi,-\psi)$ is the reflection of $\mathcal{G}(\phi,\psi)$, which ensures that the surface boundary condition (42) is satisfied.

When the reduced frequency $k$ is large, distortion by mean-flow gradients becomes more significant, and a geometric-acoustics approach is required. The phase of Eq. (43) can be perturbed inside the integral, so that the eikonal equation is satisfied at higher order in $\delta$, and $\mathcal{O}(k\delta^{2/3})$ phase distortions can be included.[16] This does not affect the total radiated power in the absence of other sources, however, and these calculations are not included here.

The final step in our analysis consists of approximating the far-field behavior of the radiation by expanding $G$ for large $r=(\phi+\psi)^{1/2}$. By the use of $r\gg|\eta_0|$, $|\tau|$ for all $\tau$ in the interval of integration, the unsteady potential can be approximated by

$$G\sim\left[D(\theta)H(\phi)/r^{\frac{1}{2}}\right]\exp\left[ik\left(1-k_3^2+\tan^2\theta\right)\phi/2\right] \quad (45)$$

where $\theta$ is the polar angle $\tan^{-1}(\psi/\phi)$ and the directivity factor $D(\theta)$ is

$$D(\theta) = \frac{2^{\frac{1}{2}}\exp(-\pi i/4)Q}{(\pi k\cos\theta)^{\frac{1}{2}}}\int_0^{\delta^{-\frac{1}{3}}\tilde{y}_s}\Delta q(\tilde{\tau})\cos(k\tau\tan\theta) \times\exp\left\{ik\left[\left(1+k_3^2+\tan^2\theta\right)\eta_0(\tilde{\tau})/2+k_n\tau\right]\right\}\mathrm{d}\tau \quad (46)$$

This is the main result of this paper, giving the $\mathcal{O}(\delta^{2/3})$ acoustic radiation in terms of a single integral involving the mean shock strength $\Delta q$ and the mean shock position $\eta_0$.

The expression (46) can be further expanded in $\delta$ by the use of $\tilde{\tau}$ and $\tau$ varying on different scales. When $\tan\theta\neq\pm k_n$, the amplitude of the integrand varies slowly compared to the phase, and integration by parts can be applied. Using also that the shock wave is attached normal to the surface, so that $(\mathrm{d}\eta_0/\mathrm{d}\tilde{\tau})(0)=0$, we obtain the expression

$$D(\theta)\sim\frac{2^{\frac{1}{2}}Qk_n\exp(\pi i/4)\Delta q(0)}{k^{\frac{3}{2}}(\pi\cos\theta)^{\frac{1}{2}}\left(k_n^2-\tan^2\theta\right)} \times\exp\left[ik\left(1+k_3^2+\tan^2\theta\right)\eta_0(0)/2\right], \qquad \tan^2\theta\neq k_n^2 \quad (47)$$

showing the directivity to be proportional to the shock strength at its base. However, Eq. (47) does not apply when one of $\pm\theta$ approaches the gust angle $\theta_g$, in which case Eq. (46) must be integrated numerically. When $\theta\to\pi/2$, on the other hand, the phase of the integrand is singular and the numerical integrator is unreliable; here the asymptotic expansion is more appropriate. For the numerical examples of Sec. IV, with the gust angles ranging from −50 to 60 deg, Eqs. (46) and (47) are applied when $\theta\leq 70$ and $\theta>70$ deg, respectively.

Finally, we note that, in addition to the integrand in Eq. (46), the solution (45) also has an unbounded phase when $\theta\to\pi/2$. Transonic theory[17] explains that receding waves cannot propagate upstream in the TSD limit but instead tend to collect in narrow regions that extend in cross-stream directions from their source, leading in this case to high-frequency oscillations near $\phi\downarrow 0$. This behavior cannot be considered physically possible arbitrarily close to $\phi=0$; in fact, higher-order terms (in $\delta$) become increasingly significant and render the governing equation locally elliptic or hyperbolic, depending on the sign of $\beta_0^2$ (Ref. 16). The unbounded phase, therefore, signifies that the linear operator $\mathcal{L}$ cannot properly account for the transition between the flows upstream and downstream of the source and that the far-field solution calculated here is not valid when $\tan\theta=\mathcal{O}(\delta^{-1/3})$.

## IV. Numerical Results

Although the problem has been formulated for a shock wave over an infinite surface, the interest for the prediction of turbomachinery noise is in shock waves over airfoils or across blade passages. However, gust–airfoil interaction encompasses a number of noise-generating mechanisms, such as the diffraction of the gust by the leading edge and the rescattering by the trailing edge. Hence, if we are to investigate the sound generation by gust–shock interaction alone, these mechanisms should be ignored, and this is achieved by neglecting the airfoil edges. Formally, we can apply the theory put forward here to a shock generated by an airfoil, or alternatively, we can argue that there exists an infinite surface that would generate the same shock.

For our numerical examples we take a shock wave calculated by the Engquist and Osher[18] variant of the Murman and Cole[19] relaxation algorithm for TSD flow. The scheme uses a backward or centered difference scheme depending on the sign of the local value of $\beta_0^2$. The shock wave is defined by a set of fixed gridpoints (Fig. 2), so that a best-fit curve must be calculated to obtain a continuous shock suitable for calculating Eq. (46). The shock wave illustrated in Fig. 2 is induced by a NACA0006 airfoil with zero angle of attack with respect to an oncoming flow of $M_\infty = 0.9$. For $b$ we take the semichord length, so that the airfoil lies between $x = 0$ and 2. The shock strength, as given by the function $-\Delta q(\tilde{y})/\delta^{2/3}$, is also indicated in Fig. 2.

We consider first the directivity patterns in the far field. The pressure is calculated directly from Eq. (7), but a direct consequence of the singular phase in Eq. (45) is that the pressure is also singular at $\phi \downarrow 0$. Therefore, we choose to represent the directivity pattern through the function $|D(\theta)|$, or through the intensity directivity,[16]

$$F(\theta) = \frac{|D(\theta)|^2 H(\phi)}{k\cos\theta} \tag{48}$$

which are both bounded. The latter represents the time-averaged energy flux across a semicircle (at large $r$), normalized by $\epsilon^2\rho_\infty U_\infty^3/2r$, and the integral

$$P = \int_0^{\pi/2} F(\theta)\,\mathrm{d}\theta \tag{49}$$

then represents the radiated acoustic power, normalized by $\epsilon^2\rho_\infty U_\infty^3 b/2$. In the far field where the flow may be considered uniform, the coordinates $\phi$ and $\psi$ agree with $x$ and $y$, and, thus, $\theta$ is also the polar angle in physical space.

A comparison between the plots of Fig. 3, showing $|D(\theta)|$ for a gust with $k = 5$, $A_t^* = 1/2$, $k_3 = 3^{-1/2}$, and varying $k_n$, confirms the conclusion drawn earlier that the sound waves are primarily propagated in the direction of the gust orientation angle $\theta_g = \tan^{-1} k_n$;

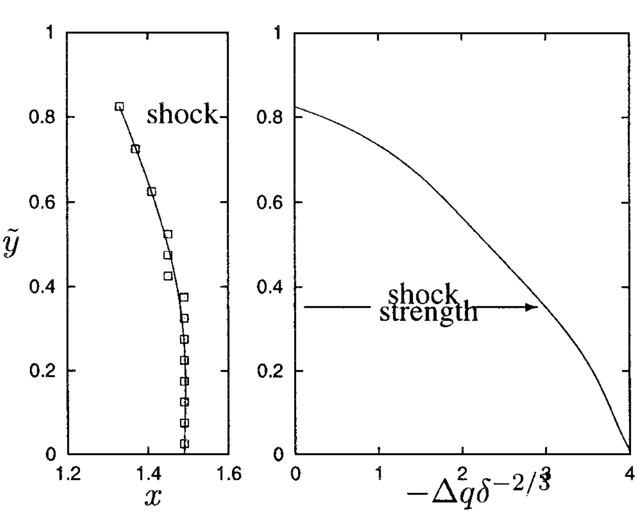


**Fig. 2 Shock position and strength for a NACA0006 airfoil at zero angle of attack with $M_\infty = 0.9$; boxes indicate the positions of the shock-wave grid points from which a best-fit curve is calculated.**

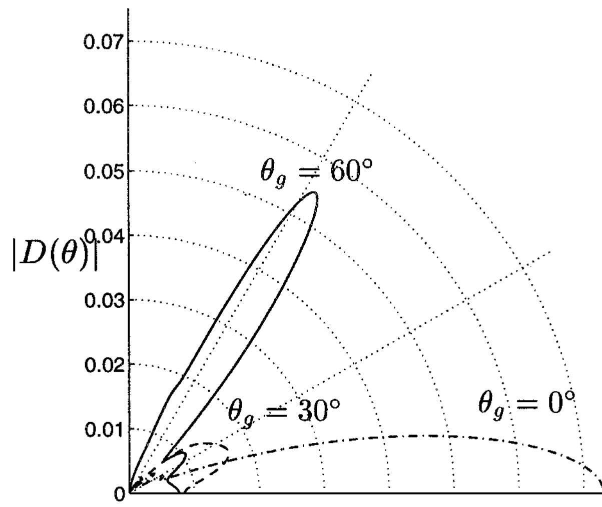


**Fig. 3 Directivity pattern for the shock wave of Fig. 2 for varying gust angles; relevant gust parameters are $k = 5$, $A_t^* = 1/2$, and $k_3 = 3^{-1/2}$.**

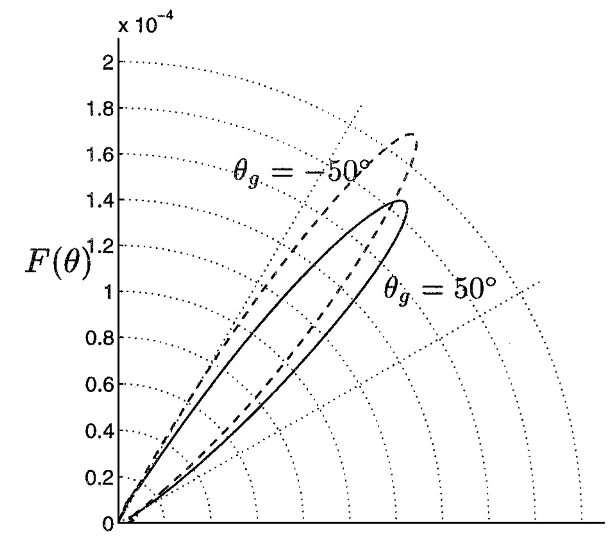


**Fig. 4 Intensity directivity pattern for the shock wave and gust of Fig. 3, but with gust angles $\theta_g = \pm$ 50 deg.**

clearly, the lobes in these graphs have maxima near the gust directions $\theta_g = 0$, 30, and 60 deg. The peaks are more pronounced for larger gust angles because the oscillations of $D(\theta)$ are more rapid for larger $\theta$. The amplitude for $\theta_g = 30$ deg is smaller than the amplitudes for $\theta_g = 0$ and 60 deg because the gust orientation here is closer to the critical angle $\Theta_g = 45$ deg for which $Q$ disappears [see Eq. (38)], corresponding in this case to $\theta_g \approx \pm 39.2$ deg.

The two cases $\theta_g = \pm 50$ deg are compared in Fig. 4. We use the function $F(\theta)$, which is proportional to $|D(\theta)|^2$ and, therefore, accentuates the peaks at $\theta = |\theta_g|$. Note that in both cases the radiation propagates in the direction $\theta \approx 50$ deg, but that the lobe for $\theta_g = -50$ deg is larger than that for $\theta_g = 50$ deg. This and similar results suggest that a gust oriented toward the surface leads to more significant interaction with the shock than a gust oriented away from the surface. Also, consistent with Fig. 4, the peak for negative gust orientation is generally found at a slightly larger observer angle than for positive gust orientation.

The directivity patterns of Fig. 5 show the influence of varying the reduced frequency $k$. Clearly, for increasing $k$, the fluctuations (with $\theta$) increase and the amplitude decreases, causing a significant decrease in radiated power, as evident also in Fig. 6. Figure 7 shows the response to varying $k_3$. The expressions for $D(\theta)$ and its asymptotic expansion, Eqs. (46) and (47), respectively, show that the acoustic power depends on $k_3$ through the coefficient $Q$ and, for observer angles near the gust angle $\theta_g$, through the exponent in Eq. (46) when there is a nonzero shock slope, that is, when $\mathrm{d}\eta_0/\mathrm{d}\tilde{\tau} \neq 0$. Note that

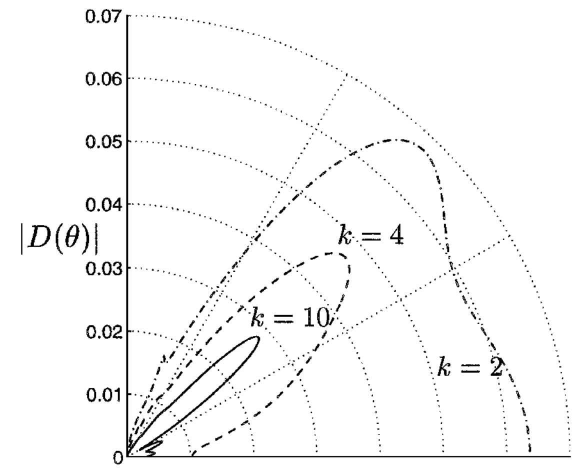


**Fig. 5** $A_t^* = 2^{-1/2}$, $\theta_g = 45$ **deg,** $k_3 = 1$**, and** $k$ **is varied.**

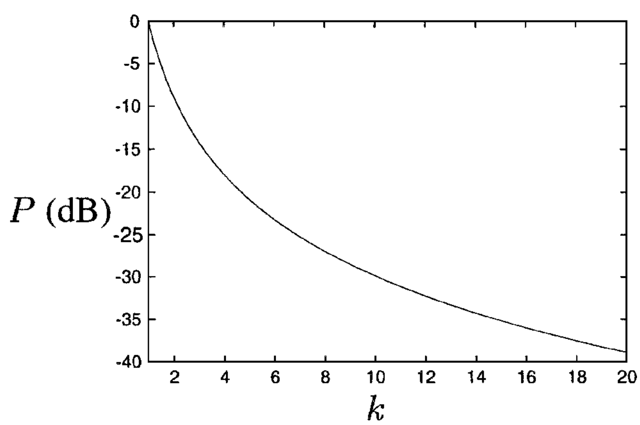


**Fig. 6 Radiated power vs** $k$ **(normalized with respect to the case** $k = 1$**) for the same shock wave and for a gust with** $A_t^* = 1/2$, $\theta_g = 30$ **deg, and** $k_3 = 1$**.**

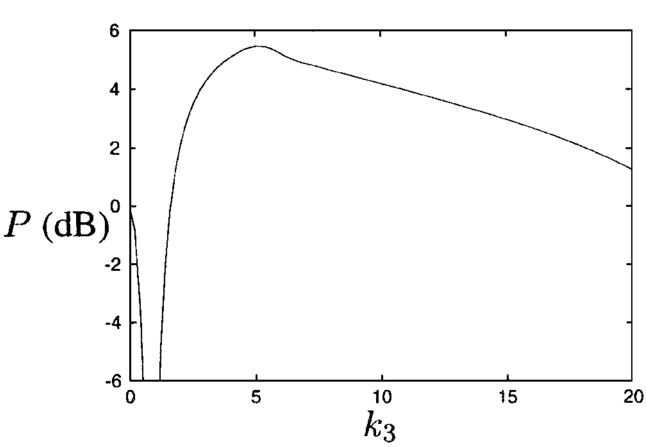


**Fig. 7 Radiated power vs** $k_3$ **(normalized with respect to the power for** $k_3 = 0$**) for the gust of Fig. 6, but with** $k = 5$**.**

although in subsonic flow there is a maximum value of the spanwise wave number above which there is no radiation, this is not the case in the transonic limit, where radiation is produced for all $k_3$.

A comparison between the acoustic radiation due to gust–shock interaction and that due to other mechanisms in a cascade is relevant in the context of turbomachinery noise. The related study of gust–airfoil interaction[16] uses the high-frequency approximation $k \gg 1$ in addition to the small-disturbance approximation $\delta \ll 1$, providing a means of determining the relative importance of the various radiation components in the limit of large reduced frequency. As in subsonic flow, the unsteady potential for the scattering of the gust by the airfoil's leading edge is $\mathcal{O}(k^{-3/2})$ at leading order. For most observer angles, it is found that the higher-order terms due to volume sources and other mean-flow effects are a factor $\mathcal{O}(k^{1/3}\delta^{2/3})$ smaller. By contrast, the potential due to gust–shock interaction that is calculated here is $\mathcal{O}(k^{-3/2}\delta^{2/3})$, a factor $\mathcal{O}(\delta^{2/3})$ smaller than the leading-edge radiation. In both cases, there is an additional factor $k^{1/2}$ when we consider observer angles near the gust orientation angle, and it follows that mean-flow non-uniformities (at the shock wave or at the leading edge) can be the dominant source of radiation, depending on the relative magnitudes of $\delta$ and $k$. For high reduced frequency and small shock strength, the large mean-flow gradients at the leading edge have a greater influence on sound generation than the relatively small discontinuity at the shock. On the other hand, the shock wave will be more important for higher Mach numbers when its strength is greater, or, indeed, for more moderate frequencies when the length scale of the leading-edge singularity is too small for significant interaction with the gust. A numerical comparison in a more comprehensive model that includes both interactions will assess the relative importance for specific gust frequencies and shock strengths.

## V. Conclusions

In general, the disturbances downstream of a shock cannot be calculated given just the mean flow and the disturbances upstream of the shock. Because of the additional degree of freedom that results from the possible shock motion, there is always one more unknown flow quantity than there are jump relations. The answer is that the complete downstream unsteady flow must be solved for simultaneously with the jump relations. For uniform mean flow, the splitting theorem is a necessary tool, whereas for TSD flow Goldstein's[1] decomposition, together with the divergence-free condition for the convected disturbances in the far field downstream, can be applied.

Useful analytical results are made possible by the small-disturbance approximation of the mean flow. In TSD flow the shock wave is long relative to the $\mathcal{O}(1)$ or smaller gust wavelength. Consequently, there are different scalings that lead to qualitatively different descriptions of the sound generated at the shock. On the gust wavelength scale at $\mathcal{O}(1)$ distances from the shock, the acoustic field may be regarded as a band of plane waves extending downstream at the gust angle. On the shock length scale, by contrast, the waves of varying magnitude interact to produce a more complicated acoustic field that resembles a point source when observed from a far. The result of these calculations is a straightforward representation of the total acoustic radiation into the far field, and a numerical investigation indicates a strong dependence on the gust angle.

The analysis shows that there are two parts to the sound generation. The waves with amplitude $\Gamma_{s2}^{(2)}$ [see Eq. (39)] are directly related to the jump in the streamwise component of the convected gust $\tilde{v}_t'$. The waves with amplitude $\Gamma_{p2}^{(2)}$, on the other hand, are directly related to the convected waves caused by the volume sources [see Eq. (36)]. The hydrodynamic motion that is sustained throughout the flow by the volume sources disappear proportional to $q$ in the far field, though its discontinuity leads to the generation of more sound waves at the shock.

The results presented here may be applied with some modification to a cascade configuration. Given the steady flow through a cascade as computed by a suitable numerical code, much of the current analysis can be applied to shock waves across the blade passages. The starting point is solution (43) for the sound generated by the gust–shock interaction without taking into account additional boundary conditions. For TSD theory, the passage flow is quasi one dimensional, so that $\Delta q$ and $\eta_0$ can be considered constant (rather than slowly varying), and these values can be taken directly from the steady solution. However, before the far-field radiation can be calculated, the flow in the blade passage must be correctly accounted for (i.e., there are now two boundary conditions) and other sources, such as rescattering at the trailing edge, must be included.

Any adaptations required to describe the gust–shock interaction when the shock is not weak are by no means trivial, because much of the analysis presented here relies on the TSD assumption. Analytical progress may be made for gust–shock interaction at high frequency, in which case previous results for shocks in uniform flow can be applied locally on the scale of the gust wavelength.

## Appendix: Uniqueness of the Decomposition

The issue of uniqueness of the unsteady flow downstream of the shock wave is most easily examined by considering the shock-wave problem at leading order. The solution is trivial, but by ignoring the usual divergence-free relation (22), it clarifies the dilemma that is faced at higher order. Instead of Eq. (29), or rather $\mathcal{L}(G^{(0)}) = 0$ using the notation of Sec. III, the governing equation is

$$\mathcal{L}\left(G^{(0)}\right) = -ik\left(\bar{h}_a^{(0)} + \bar{h}_d^{(0)} + k_n\bar{h}_b^{(0)} + k_3\bar{h}_c^{(0)}\right)\exp[ik(\phi + k_n\psi)]$$

giving us the particular solution

$$G_p^{(0)} = \frac{i\left(\bar{h}_a^{(0)} + \bar{h}_d^{(0)} + k_n\bar{h}_b^{(0)} + k_3\bar{h}_c^{(0)}\right)}{k\left(1 + k_n^2 + k_3^2\right)}\exp[ik(\phi + k_n\psi)] \quad \text{(A1)}$$

a potential function that represents convected disturbances, that does not contribute to the pressure field, and that renders $\nabla G_p^{\prime(0)} + \boldsymbol{v}^{\prime(0)}$ divergence free. A complementary solution $G_c^{(0)}$, which reduces to Eq. (31) when Eq. (23) is imposed, cancels the velocity normal to the surface, and for the remaining component of $G^{(0)}$ we look for a solution of the form

$$G_s^{(0)} = \Gamma_2^{(0)}\exp\left\{ik\left[\left(1 - k_n^2 - k_3^2\right)\phi/2 + k_n\psi\right]\right\} \quad \text{(A2)}$$

for some constant $\Gamma_2^{(0)}$. At $\mathcal{O}(\epsilon\delta^0)$ the five jump relations then give

$$\Gamma_2^{(0)} = 0, \qquad \bar{h}_{b2}^{(0)} = A_n + k_n\left(\bar{h}_{a2}^{(0)} - A_t^*\right)$$

$$\bar{h}_{c2}^{(0)} = A_3 + k_3\left(\bar{h}_{a2}^{(0)} - A_t^*\right), \qquad \bar{h}_{d2}^{(0)} = B \quad \text{(A3)}$$

and an expression for the time derivative of $\eta^{\prime(0)}$.

Although the only possible downstream solution in the limit $\delta \to 0$ is the flow that we also find upstream, relations (52) indicate an unresolved degree of freedom that is attributed to decomposition (3) not being uniquely defined. Accordingly, the total unsteady velocity $\boldsymbol{u}'$ and the unsteady pressure $p'$ are independent of $\bar{h}_{a2}^{(0)}$. Any choice for this remaining unknown will suffice, and it is natural to take $\bar{h}_{a2}^{(0)} = \bar{h}_{a1}^{(0)} = A_t^*$ for a number of reasons. First, this decomposition is consistent with the decomposition upstream of the shock because the same expressions for $G^{(0)}$ and $\boldsymbol{v}'$ are then valid on both sides of the shock. Second, this choice is consistent with the classical splitting theorem,[10] which is valid when the mean flow is uniform. Using Helmholtz's theorem (see Arfken,[20] for example), the splitting theorem in this context argues the existence of a (unique) decomposition of $\boldsymbol{u}'$ into $\nabla G' + \boldsymbol{v}'$ such that $\nabla\cdot\boldsymbol{v}' = 0$. For our choice of $\bar{h}_{2a}^{(0)}$ at this order in $\delta$, the corresponding velocity $\boldsymbol{v}'$ satisfies that condition. Finally, for the upstream flow, the unique decomposition implied by the splitting theorem is also achieved by the far-field boundary condition[1] $\lim_{\phi\to-\infty} G' = 0$. Although the same condition cannot be applied downstream at $\phi \to +\infty$ in general, we can state that with our choice of $\bar{h}_{2a}^{(0)}$ the only components of $G'$ that do not disappear with the mean-flow distortion solve the homogeneous form of the convected wave equation (6).

The ambiguity described here is less transparent in the analysis at higher order, and we must, therefore, generalize our choice for $\bar{h}_{2a}^{(0)}$ to higher orders. In addition to imposing the usual relation $\nabla\cdot\boldsymbol{v}' = 0$ for uniform mean-flow conditions far upstream, we must impose the same relation for uniform mean-flow conditions downstream of the shock, giving us relation (23). The point to emphasize is that this relation is not essential for the solution of the problem at hand, though it ensures that decomposition (3) is unique, and the mathematics are more easily managed when calculating the conditions across the shock. Particularly downstream of the shock, where the usual assumption $\lim_{\phi\to-\infty} G' = 0$ cannot give solace, it provides the obvious choice for the one remaining (arbitrary) unknown.

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## References

[1] Goldstein, M. E., "Unsteady Vortical and Entropic Disturbances of Potential Flows Round Arbitrary Obstacles," *Journal of Fluid Mechanics*, Vol. 89, 1978, pp. 433–468.

[2] Ribner, H. S., "Convection of a Pattern of Vorticity Through a Shock Wave," NACA TN 2864, Jan. 1953.

[3] Moore, F. K., "Unsteady Oblique Interaction of a Shock Wave with a Plane Disturbance," NACA TR 1165, Jan. 1954.

[4] Hardy, B., and Atassi, H. M., "Interaction of Acoustic, Entropic, and Vortical Waves with a Plane Shock," AIAA Paper 97-1614, 1997.

[5] Ribner, H. S., "Spectra of Noise and Amplified Turbulence Emanating from Shock–Turbulence Interaction," *AIAA Journal*, Vol. 25, No. 3, 1987, pp. 436–442.

[6] Mahesh, K., Lee, S., Lele, S. K., and Moin, P., "The Interaction of an Isotropic Field of Acoustic Waves with a Shock Wave," *Journal of Fluid Mechanics*, Vol. 300, 1995, pp. 383–407.

[7] Williams, M. H., "Unsteady Thin Airfoil Theory for Transonic Flows with Embedded Shocks," *AIAA Journal*, Vol. 18, No. 6, 1980, pp. 615–624.

[8] Verdon, J. M., and Caspar, J. R., "A Linearized Unsteady Aerodynamic Analysis for Transonic Cascades," *Journal of Fluid Mechanics*, Vol. 149, 1984, pp. 403–429.

[9] Rusak, Z., Giddings, T. E., and Cole, J. D., "Interaction of a Weak Shock with Freestream Disturbances," *AIAA Journal*, Vol. 33, No. 6, 1995, pp. 977–984.

[10] Goldstein, M. E., *Aeroacoustics*, McGraw–Hill, New York, 1976, pp. 220–222.

[11] Cole, J. D., and Cook, L. P., *Transonic Aerodynamics*, North-Holland, Amsterdam, 1986, pp. 46–56.

[12] Kerschen, E. J., and Balsa, T. F., "Transformation of the Equation Governing Disturbances of a Two-Dimensional Compressible Flow," *AIAA Journal*, Vol. 19, No. 10, 1981, pp. 1367–1370.

[13] Thompson, P. A., *Compressible-Fluid Dynamics*, McGraw–Hill, New York, 1972, p. 323.

[14] Kerschen, E. J., and Myers, M. R., "Perfect Gas Effects in Compressible Rapid Distortion Theory," *AIAA Journal*, Vol. 25, No. 3, 1987, pp. 504–507.

[15] Cannon, J. R., *The One-Dimensional Heat Equation*, Addison Wesley Longman, Reading, MA, 1984, pp. 54, 55.

[16] Evers, I., and Peake, N., "Noise Generation by High-Frequency Gusts Interacting with an Airfoil in Transonic Flow," *Journal of Fluid Mechanics*, Vol. 411, 2000, pp. 91–130.

[17] Landahl, M. T., *Unsteady Transonic Flow*, Cambridge Univ. Press, Cambridge, England, U.K., 1989, pp. 9–14.

[18] Engquist, B., and Osher, S., "Stable and Entropy Satisfying Approximations for Transonic Flow Calculations," *Mathematics of Computation*, Vol. 34, No. 149, 1980, pp. 45–75.

[19] Murman, E. M., and Cole, J. D., "Calculation of Plane Steady Transonic Flows," *AIAA Journal*, Vol. 9, No. 1, 1971, pp. 114–121.

[20] Arfken, G., *Mathematical Methods for Physicists*, 2nd ed., Academic Press, New York, 1970, p. 67.

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